7 The PAMO in Ouagadougou: July 2001

PAMO in Ouagadougou 2001: Day 1
Time: 4.5 hours

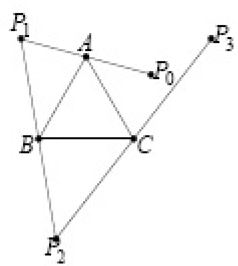
1. Find all integers $n \geqslant 1$ such that $\frac{n^3+3}{n^2+7}$ is an integer.

2. Let $n \geqslant 1$ be an integer. A child builds a wall along a line with n identical cubes. He lays the first cube on the line and at each subsequent step, he lays the next cube either on the ground or on top of another cube so that it has a common face with the previous one.

   Let u_n be the number of such distinct walls.

   1. Find u_1, u_2, u_3 and u_4.
   2. Determine u_n in terms of n.

3. Let ABC be an equilateral triangle and let P_0 be a point which is outside this triangle such that triangle AP_0C is isosceles with a right angle at P. We set $AP_0 = a$.



   A fly starts from point P_0 and turns around triangle ABC as follows:

   From P_0 the fly goes to P_1 which is the symmetric point of P_0 with respect to A. From P_1 it goes to P_2 which is the symmetric point of P_1 with respect to B and then to P_3 the symmetric point of P_2 with respect to C, then to P_3 with respect to A and so on. Compare the distances P_0P_1 and P_0P_n for all n.

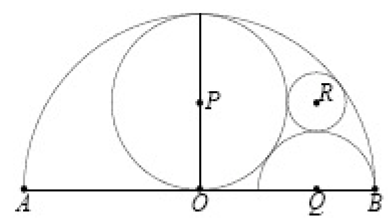
PAMO in Ouagadougou 2001: Day 2
Time: 4.5 hours

4. Let $n \geqslant 1$ be an integer and $a > 0$ a real number. Find the number of solutions $(x_1, \ldots, x_n)$ of the equation
$$\sum_{i=1}^{i=n}(x_i^2 + (a - x_i)^2) = na^2$$
such that x_i belong to the interval $[0, a]$, for $i = 1, 2, \ldots, n$.

5. Let $[x]$ denote the greatest integer less than or equal to x. Calculate
$$\left[\sqrt{1}\right] + \left[\sqrt{2}\right] + \cdots + \left[\sqrt{2001}\right].$$

6. $S1$ is a semicircle with centre O and diameter AB. A circle $C1$ with centre P is drawn tangent at O to AB and tangent to $S1$. A semicircle $S2$ is drawn with centre Q on AB, tangent to $C1$ and to $S1$. A circle $C2$ with centre R is drawn, internally tangent to $S1$ and externally tangent to $S2$ and $C1$.

   Prove that $OPRQ$ is a rectangle.



PAMO in Ouagadougou 2001: Day 1
Solutions

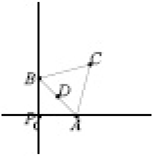
1.
$$\frac{n^3+3}{n^2+7} - n - \frac{7n-3}{n^2+7} \in \mathbb{Z} \iff \frac{7n-3}{n^2+7} \in \mathbb{Z}$$

If $n \geqslant 7$ then $0 < 7n-3 < 7n+7 \leqslant n^2+7$ and so there are no solutions for $n \geqslant 7$. We then check the remaining cases and find that the only solution are $n = 2$ and $n = 5$.

2. Note that it is impossible to build a wall with gaps in it, but any wall with no gaps can be built. We choose a unique method for building each wall: build the columns one at a time, going from left to right. For every block after the first, the child has two choices, namely to continue stacking blocks in the current column or to start a new column. Thus there are 2^{n-1} ways to build a wall in this way and so there are 2^{n-1} unique walls.

   Thus $u_1 = 1$, $u_2 = 2$, $u_3 = 4$, $u_4 = 8$ and $u_n = 2^{n-1}$.

3. We put this into co-ordinate geometry with P_0 at the origin, A at $(1,0)$ and B at $(0,1)$. Let D be the midpoint of AB, namely $(\frac{1}{2}, \frac{1}{2})$. Then $CD = \sqrt{3} \cdot AD = \sqrt{3} \cdot \frac{\sqrt{2}}{2}$. But C lies on the line $x = y$ so $C = (\frac{\sqrt{3}+1}{2}, \frac{\sqrt{3}+1}{2})$.



   Now

$$\begin{aligned}
P_1 &= A + (A - P_0) = (2,0) \\
P_2 &= C + (C - P_1) = (\sqrt{3}-1, \sqrt{3}+1) \\
P_3 &= B + (B - P_2) = (1-\sqrt{3}, 1-\sqrt{3}) \\
P_4 &= A + (A - P_0) = (1+\sqrt{3}, \sqrt{3}-1) \\
P_5 &= C + (C - P_4) = (0,2) \\
P_6 &= B + (B - P_5) = (0,0)
\end{aligned}$$

so $P_6 = P_0$ and so the points cycle with period 6. So using the standard distance formula,

$$\frac{P_0P_n}{P_0P_1} = \begin{cases} 0 & \text{for } n \equiv_6 0 \\ 1 & \text{for } n \equiv_6 1 \\ \sqrt{2} & \text{for } n \equiv_6 2 \\ \sqrt{2-\sqrt{3}} & \text{for } n \equiv_6 3 \\ \sqrt{2} & \text{for } n \equiv_6 4 \\ 1 & \text{for } n \equiv_6 5 \end{cases}$$

PAMO in Ouagadougou 2001: Day 2
Solutions

4.

$$\begin{aligned} na^2 &= \sum_{i=1}^{n} (x_i^2 + (a-x_i)^2) \\ &= 2\sum_{i=1}^{n} x_i^2 + na^2 - 2a\sum_{i=1}^{n} x_i \\ \implies \quad 2\sum x_i^2 &= 2a\sum x_i \\ \implies \quad \sum x_i(x_i - a) &= 0. \end{aligned}$$

But $0 \leqslant x_i \leqslant a \quad \forall i$ so $x_i(x_i - a) \leqslant 0 \quad \forall i$ and their sum is 0, so each time must be zero. So either $x_i = 0$ or $x_i = a$ for each i.

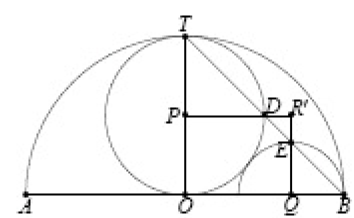
For each x_i there are 2 choices, so there are 2^n solutions.

5. Between n^2 and $(n+1)^2$ there are $2n$ numbers. Including n^2 itself we have $2n+1$ and so the sum up to the term $\sqrt{(n+1)^2 - 1}$ is $\sum_{i=1}^{n} i(2i+1)$ since each i is added $2i+1$ times.

The largest square less than 2001 is $44^2 = 1936$. So the answer is

$$\begin{aligned} \sum_{i=1}^{43} i(2i+1) + 44\cdot 66 &= 2\sum_{i=1}^{43} i^2 + \sum_{i=1}^{43} i + 44\cdot 66 \\ &= 2\cdot \tfrac{43.44.87}{3} + 43\cdot 22 + 44\cdot 66 \\ &= 43\cdot 44\cdot 29 + 43\cdot 22 + 44\cdot 66 \\ &= 58718 \end{aligned}$$

6. Construct BT where T is the point of tangency of C_1 and S_1. Let this line intersect C_1 at D and S_2 at E.



We will now prove that D is on PR and E on RQ, and that $PD \perp PO$ and $EQ \perp OQ$. Firstly $\angle B = \angle T = 45^\circ$ and $PO \perp OB$ (radius perpendicular to tangent). In S_2, $EQ = QB$ (radii) and so $\angle BEQ = \angle EBQ = 45^\circ$ and $EQ \perp QB$. Similarly in C_1, $PD = PT$ and so $PD \perp PO$.

Now to prove D on PR and E on RQ we will have to show that R', the intersection of PD and EQ is the same as R. To do this it suffices to show that the distances from R' to S_1, D and E are the same.

Now $R'QOP$ is a rectangle (4 right angles) and the line DE cuts it at 45° to the sides, so $DR' = ER'$. The distance from R' to S_1 is

$$\begin{aligned} OT - OR' &= 2PO - PQ \qquad \text{(diagonals of rectangle)} \\ &= 2PO - (PO + EQ) \qquad \text{(radii)} \\ &= PO - EQ \\ &= QR' - EQ \\ &= R'E \end{aligned}$$

as desired. So R' is R and thus $POQR$ is a rectange.